Chapter 3: Applications of derivatives

Section 3.1: Extrema on an Interval

The largest and smallest values of a function on a particular interval are known as *extrema* (singular *extremum*). Extrema can be divided into two categories: *absolute* and *relative* (also referred to as *global* and *local*, respectively).

Absolute (or global) *extrema* are the points at which the function is larger or smaller than at any other on a *closed interval*. To be more precise:

- if $f(c) \geq f(x)$ for all x on the closed interval $[a, b]$, then $f(c)$ is an absolute maximum
- if $f(c) \leq f(x)$ for all x on the closed interval $[a, b]$ interval, then $f(c)$ is an absolute minimum

If $f(c)$ is an absolute maximum, you can also say that there is an absolute maximum *at* $x = c$ or *at* $(c, f(c))$. Every closed interval has at least one absolute maximum and minimum (Theorem 8.1), but there can be more than one. On a horizontal line, for example, *every* x value would be an absolute maximum *and* an absolute minimum.

Relative extrema are the largest or smallest values on a particular *open interval*. There cannot be a relative extrema at the endpoint of an open interval interval because an open interval does not *contain* its endpoints. Therefore, if relative extrema exist on a particular interval, they must be located somewhere in the middle. Think of relative extrema as the "hills" or "valleys" of a graph.

For example, $f(x) = x^2$ has a relative minimum at $x = 0$ on [-3, 3], because [-3, 3] contains the open interval (3, -3), which contains 0, at which point the y-value is less than that of any other y-value on the interval.

$f(x)$ cannot be said to have not have a relative minimum on the intervals (-3,0) or (0, 3) because those intervals do not contain $x = 0$. However, it does have an *absolute* minimum at $x = 0$ on the intervals [-3, 0] and [0, 3].

A relative extremum may also be an absolute extremum and vice versa, but not all relative extrema are absolute extrema, and not all absolute extrema are relative extrema.

Critical numbers

A *critical number* is any x-value c such that such that $f'(c) = 0$ or $f'(c)$ does not exist. Critical numbers are important, because *all relative extrema occur at critical points* (Theorem 8.2) and *all absolute extrema occur at either critical points or endpoints*.

To find the absolute extrema of a function over an interval, one must evaluate the function at each critical point and endpoint. The point (or points) at which the largest value occurs is the absolute maximum and the point (or points) at which the smallest value occurs is the absolute minimum.

The converse of Theorem 8.2 is not necessarily true: n*ot all critical points are relative extrema.* For example $x = 0$ is a critical point of $f(x) = x^3$, because $f'(0) = 3(0)^2 = 0$, but x^3 does not have an extremum at $x = 0$.

Section 3.2: Rolle's theorem and the Mean Value Theorem

Rolle's theorem: If f is continuous over $[a, b]$, differentiable over (a, b), and $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

-In other words, if the function ends up where it started, it must have changed direction.

Mean value theorem If f is continuous over $[a, b]$, and differentiable over (a, b) then there is at least one number c on (a, b) at which that the instantaneous slope equals the average slope.

-This means that if my average speed on the way to work this morning was 50mph, then there must have been at least one moment when the my speed was *exactly* 50mph.

Notice that these theorems, although they confirm the existence of certain points, do not tell you how to *find* the points at which these things occur. The practical utility of these theorems, therefore, is somewhat limited. However, they will be used to prove other important theorems later on. In particular, the fundamental theorem of calculus can be proven by using the mean value theorem.

Section 3.3: Increasing and Decreasing Functions and the First Derivative Test

If a function is continuous on $[a, b]$ and differentiable on (a, b), then that function is said to be

- increasing on $[a, b]$ if $f'(x) > 0$, for all x in (a, b)
- decreasing on $[a, b]$ if $f'(x) < 0$, for all x in (a, b)
- constant on $[a, b]$ if $f'(x) = 0$ for all x in (a, b)

The *first derivative test* allows you to categorize critical points based on whether the function is increasing or decreasing in their vicinity:

Let c be a critical point of a function $f(x)$ that is continuous on an open interval containing c. If $f(x)$ is differentiable on that interval, except possibly at c, then

- If $f'(x)$ changes from positive to negative at c, $f(c)$ is a relative maximum
- If $f'(x)$ changes from negative to positive at c, $f(c)$ is a relative minimum
- If $f'(x)$ has the same sign on both sides of c, $f(c)$ is neither a relative maximum nor a relative minimum

To analyze a fuction using the First Derivative Test, use the following procedure:

1. Calculate the derivative
2. locate the critical numbers, by finding where the derivative = 0 or does not exist
3. Determine the sign of $f'(x)$ in the intervals between the critical numbers
4. Apply the first derivative test to classify the critical points as maxima, minima or neither

Section 3.4: Concavity and the Second Derivative Test

Concavity is defined as the sign of the second derivative.

- If $f''(a) > 0$, $f(x)$ is said to be *concave upward* at $x = a$.
- If $f''(a) < 0$, $f(x)$ is said to be *concave downward* at $x = a$.

A point where the concavity changes sign is known as an *point of inflection*. (Technically, the function must also have a horizontal tangent at this point to be considered a point of inflection, though not all textbooks require this.)

Just as all relative extrema occur at places where $f'(x) = 0$ or $f'(x)$ does not exist (i.e. critical points), all inflection points occur at places where the $f''(x) = 0$ or $f''(x)$ does not exist.

The notion of concavity provides an alternative method for classifying critical points, known as the *second derivative test*. Essentially, if f(x) has a critical point at c and the function is concave up, then $f(c)$ is a minimum; if f(x) has a critical point at c and the function is concave down, then $f(c)$ is a maximum. More precisely,

Let $f(x)$ be a function such that $f'(c) = 0$ and $f''(x)$ exists on an open interval containing c. Then,

- If $f''(c) > 0$, $f(x)$ has a relative maximum at c
- If $f''(c) < 0$, $f(x)$ has a relative minimum at c
- If $f''(c) = 0$, the second derivative test fails. Use the First Derivative Test instead.

How to analyze the motion of a particle

1. Find the roots $f(x) = 0$. These are where the position equals zero
2. Find the critical points: $f'(x) = 0$. These are where the velocity equals zero.
3. Find the inflection points $f''(x) = 0$. These are where the acceleration equals zero.